1962

Predicted transient response of complex poles by means of a residue ratio

Rameshchandra Shankerbhai Patel

PREDICTED TRANSIENT RESPONSE OF COMPLEX POLES

BY MEANS OF A RESIDUE RATIO

BY

RAMESH. S. PATEL

A

THESIS

Submitted to the faculty of the

SCHOOL OF MINES AND METALLURGY

OF THE

UNIVERSITY OF MISSOURI

In partial fulfillment of the work required for the Degree

of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

Rolla, Missouri

1961

Approved

by

Robert D. Chenoweth (Advisor) John A. Nelson

Carl F. Richards Thomas R. Faucett

ACKNOWLEDGMENTS

The author wishes to express his appreciation to Dr. R. D. Chenoweth, Professor of Electrical Engineering for his help and invaluable guidance during the course of this study and also to Dr. R. C. Harden, Associate Professor of Electrical Engineering for his interest and encouragement in the selection of this study.

The author also extends his appreciation to Mr. J. B. Peatman, Assistant Professor of Electrical Engineering for the selection of the problem and guidance.

The helpful suggestions given by Mr. E. F. Richards, Instructor of Electrical Engineering is also appreciated.

The author's thanks are also extended to Mrs. E. F. Richards for typing this thesis.

TABLE OF CONTENTS

Page

LIST OF TABLES

Page

LIST OF FIGURES

I. INTRODUCTION

Transient response plays an important role in servomechanism, electronic and electrical circuits. A time response of a linear system is the intuitive and immediately comprehensible description of the system performance. The utility of the system performance in television and high speed computer circuits can be determined by its transient characteristics. By locating certain points on the transient response curve, the designer is enabled to predict the shape of the response with some precision. The choice of the overall transfer function (or location of the system singularities on the s-plane plot) is decided by speed, accuracy and stability of the response required by the specifications. When the test signals, such as impulse, step or ramp, are applied to the input of the system, the characteristics of the system are obtained. The Laplace transformation technique is a desirable approach for the determination of the transient response.

The complex poles or repeated poles in the overall transfer function often have a predominant effect on the transient response of the system. Simple poles and zeros which are far from the imaginary axis have negligible effect. Among the important characteristics of transient response influenced by the position of poles and zeros are the initial value and the time and amplitude of the maximum value. For instance, quick acting relays will discriminate between transient impulses solely according to the height of the first current crest; their satisfactory operation therefore depends on a setting that is determined from the maximum current wave.

In this thesis a quick prediction method based on the ratio of residues is developed for determining the initial and the maximum value of the transient responses associated with either repeated or complex poles. The residue ratio can be evaluated promptly from the transfer function by root locus methods. The residue ratio is used with a normalized standard plot to indicate the initial value and the characteristic response of the transfer function to a unit impulse input. The standard plot for the complex poles or the repeated poles can be defined as a transient response which has a maximum value at the time equal to one second.

II. REVIEW OF LITERATURE

Transient response associated with complex poles and the effect of adding simple poles or zeros to the complex poles has been discussed by J. G. Truxal (1)*. The transient response of a simple pole with impulse input is merely a decaying exponential term. He has also plotted the transient response of complex poles with different damping ratios for a unit step and unit impulse input. By means of frequency normalization, complex poles which are far from the origin, can be transferred back to an undamped natural frequency, ω_n, equal to unity to study the transient response. Truxal has also shown how to obtain the approximate picture of a transient response in which the time constant of the envelope of a damped oscillation is a reciprocal of the real part of either pole i.e., the time $\frac{1}{\zeta\omega_n}$. The overshoot also varies with the damping factor and are plotted in the book. Truxal has established the time for maximum value for complex poles.

The addition of a zero to complex poles will give relatively poor stability, while the effect of adding a simple pole is, in almost every case, stabilizing. The additional term increases both time delay and the rise time in the step function response. If the additional pole is placed at least six times as far from the imaginary axis as the complex poles the response is negligible in a sense that its amplitude is small and it dies out very rapidly. When the simple pole is in the ficinity of the dominant poles, its effect in an overall system is in the amplitude and in a phase of the damped oscillation term. The addition of a dipole will affect the response to a step function only slightly. Hence

*All references appear in the Bibliography.

as a result of the feasibility of neglecting poles far from an imaginary axis and dipoles, etc., concluding remarks can be made that the majority of the pole-zero configurations encountered in the design of the feedback control systems can be approximated by two or three poles and one or two finite zeros.

Skilling (2) has given the equation $\omega t = \tan^{-1}\frac{\omega}{a}$ for the time at which the maximum value occurs. Where ω equals the imaginary part of the complex pole and a equals the real part of the complex poles. Once the time for maximum value is known then the maximum value is evaluated by substituting that time in the response function. In the case of small damping the general appearance of the transient response is a sine wave. The value of 'a' is small, therefore, the fraction, $\frac{\omega}{a}$, is not very different from ∞ and the time of maximum value, like the time of tangency to the envelope, occurs at the first quarter cycle. In the case of large damping the general appearance of a transient response will be a distorted sinusoidal wave-form and with correspondingly large 'a'; $\frac{\omega}{a}$ has a positive value and the maximum value will precede the quarter cycle.

Mulligan (3) has taken the overall transfer function and plotted its transient response for a unit step and indicated the promiment features of the response, e.g. rise time, delay time, settling time, etc. Then he divides the transient response into three stages. The singularities which are far from the imaginary axis, or which determine a high frequency response have an effect only in an initial stage of the response, i.e. in range I. Those singularities which are near to the origin will effect the steady state value of the response, i.e. range III. In between the two is range II which is the most important portion in the response.

Mulligan (3) and Kuo (4) both have used the name dominant poles which make a response effective and give rise to terms with the large time constant and large amplitudes. Also the name auxiliary poles was given to those poles which are far from the imaginary axis. Mulligan also states an effect of the real pole and zero. Effect of a zero is to increase the speed of the response as well as the overshoot for step input, while the addition of a pole has the inverse effect.

III. ANALYTIC METHOD FOR CALCULATING THE TRANSIENT RESPONSE

(A) REPEATED POLES

Consider the transfer function:

$$G(s) = \frac{K N(s)}{(s+p_1)^2 D(s)} \tag{3-1}$$

$$= \frac{A}{s+p_1} + \frac{B}{(s+p_1)^2} + \frac{N'(s)}{D(s)} \tag{3-2}$$

Considering only the repeated poles and normalizing a residue term with B with a pole p_1. Such that $B = p_1 x b$. Therefore,

$$G'(s) = \frac{A}{s+p_1} + \frac{b \cdot p_1}{(s+p_1)^2} \tag{3-3}$$

Taking the inverse of the transform of Eq. (3-3),

$$g'(t) = \left[A e^{-p_1 t} + b \cdot p_1 \cdot t e^{-p_1 t}\right] u(t) \tag{3-4}$$

Letting $p_1 = \frac{1}{T}$, substituting in Eq. (3-4) and simplifying

$$g'(t) = \left(A + b\frac{t}{T}\right) e^{-\frac{t}{T}} u(t) \tag{3-5}$$

$$g'(t) = b e^{\frac{A}{b}} \left[\frac{t + A/_bT}{T}\right] e^{-\left[\frac{t + A/_bT}{T}\right]} u(t) \tag{3-6}$$

Letting $\tau = t + \frac{A}{b}T$

$$g'\left(\tau - \frac{A}{b}T\right) = b e^{\frac{A}{b}} \frac{\tau}{T} e^{-\frac{\tau}{T}} u\left(\tau - \frac{A}{b}T\right) \tag{3-7}$$

The transient response described in Eq. (3-7) is shown in Fig. 3-1.

Equation (3-7) will give the initial value at the time $\tau_i = \frac{A}{b}T$. Differentiating Eq. (3-7) and equating to zero for evaluating time, τ, for maximum value

$$\frac{d g'(\tau - \frac{A}{b}T)}{d\tau} = \frac{b}{T} e^{\frac{A}{b}} \left[1 - \frac{\tau}{T}\right] e^{-\frac{\tau}{T}} = 0 \tag{3-8}$$

$$\therefore \tau_m = T \tag{3-9}$$

Hence the time, τ_m, for maximum value is the inverse of the value of the double pole.

Again rearranging Eq. (3-4)

$$g'\left(\frac{t}{T}\right) = A e^{-\frac{t}{T}} + b \frac{t}{T} e^{-\frac{t}{T}}$$

$$\therefore g'\left(\frac{t}{T}\right) = b e^{\frac{A}{b}} \left(\frac{t}{T} + \frac{A}{b}\right) e^{-\left(\frac{t}{T} + \frac{A}{b}\right)} u\left(\frac{t}{T}\right) \tag{3-10}$$

Let $\tau_n = \frac{t}{T} + \frac{A}{b}$

(3-11)

$$\therefore \frac{t}{T} = \tau_n - \frac{A}{b} \tag{3-12}$$

Substituting Eq. (3-11) and Eq. (3-12) in Eq. (3-10):

$$g'\left(\tau_n - \frac{A}{b}\right) = b e^{\frac{A}{b}} \tau_n e^{-\tau_n} u\left(\tau_n - \frac{A}{b}\right) \tag{3-13}$$

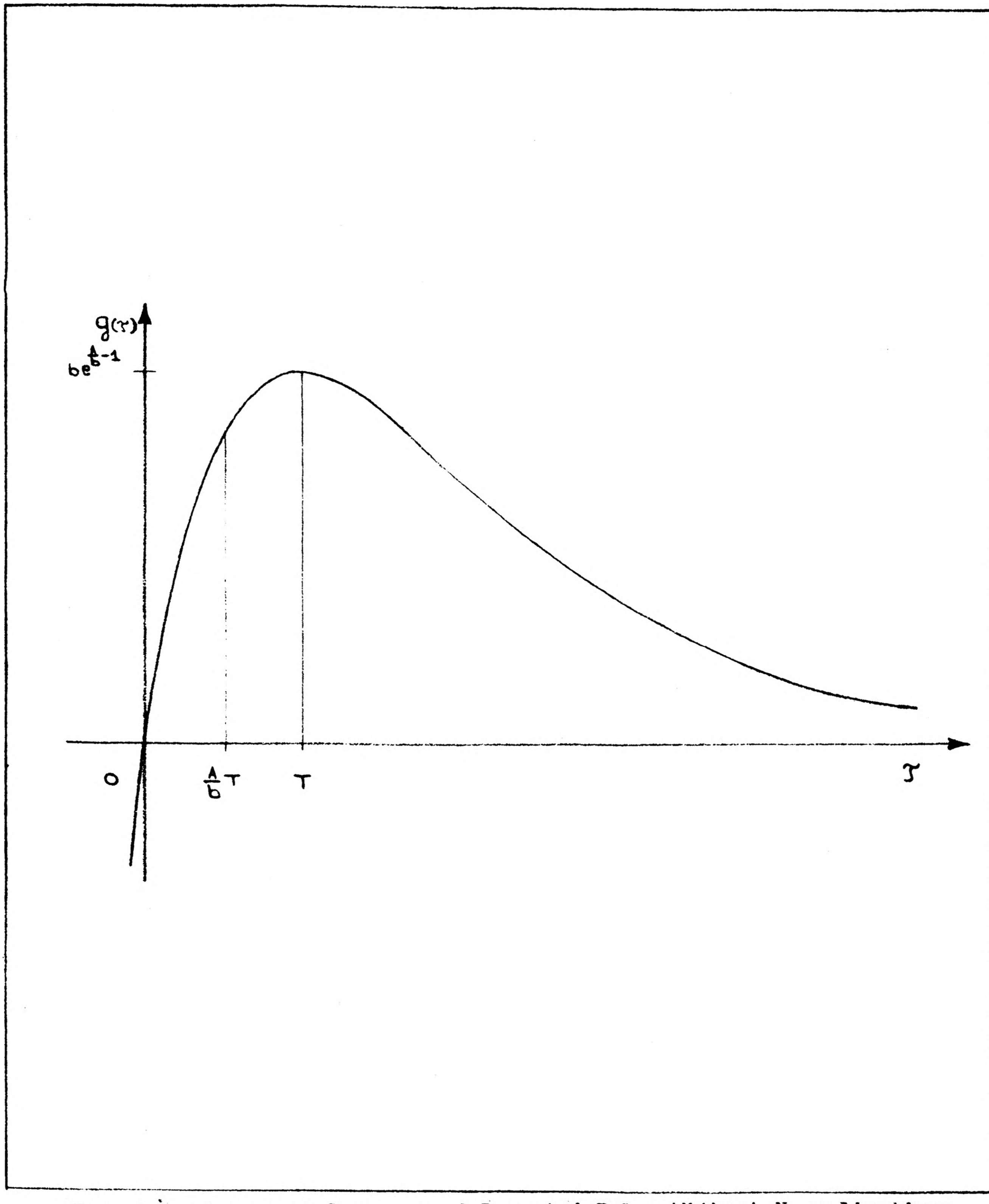

Fig. 3-1 Transient Response of Repeated Poles Without Normalization

Choosing the magnitude normalization factor $\beta = \frac{1}{b e^{\frac{A}{b}}}$ and multiplying Eq. (3-13) by β

$$g''(\tau_n - \tfrac{A}{b}) = e\, \tau_n\, e^{-\tau_n}\, u(\tau_n - \tfrac{A}{b}) \tag{3-14}$$

Equation (3-14) is in the normalized form where the maximum value is at τ_n= 1 sec., maximum amplitude is 1 and the initial value will depend upon the residue ratio $\frac{A}{b}$. The standard response (Eq. 3-14) is shown in Fig. 3-2.

Eq. (3-14) is the standard curve or normalization curve for repeated poles.

Thus the standard curve for repeated poles is defined by the response to repeated poles whose time for maximum value is at 1 sec. and whose maximum amplitude is 1. The time on this curve corresponding to the initial value is dependent on the residue ratio $\frac{A}{b}$.

(B) COMPLEX POLES

Consider the transfer function:

$$G(s) = \frac{K(s+Z_1)(s+Z_2)}{s(s+P_1)(s+P_2)[(s+a)^2+\omega^2]} \tag{3-14}$$

Which can be arranged as:

$$G(s) = \frac{K\,N(s)}{[(s+a)^2+\omega^2]\,D(s)} \tag{3-15}$$

Expanding by means of Heaviside's theorem:

$$G(s) = \frac{A(s+a)}{(s+a)^2+\omega^2} + \frac{C\cdot\omega}{(s+a)^2+\omega^2} + \frac{N'(s)}{D'(s)} \tag{3-16}$$

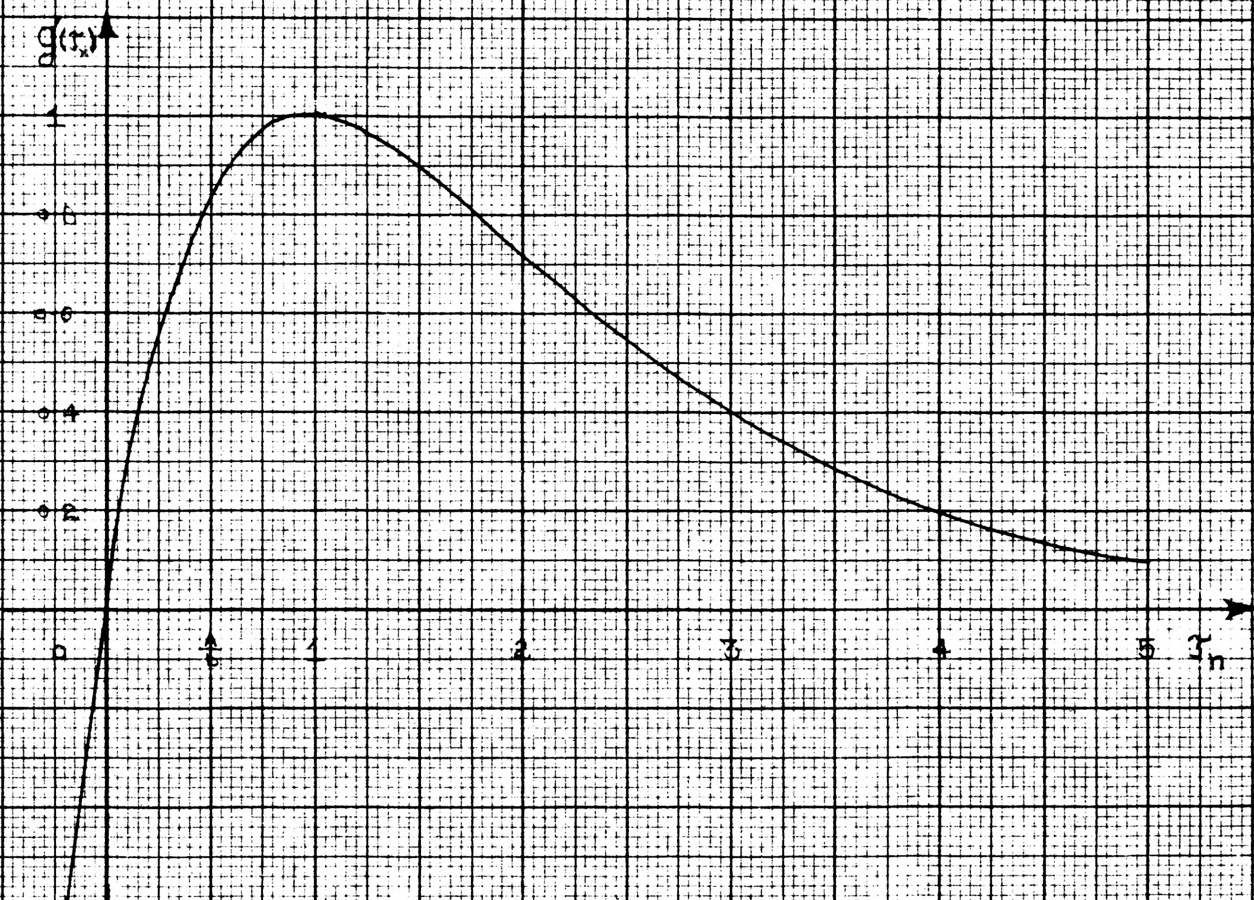

Fig. 3-2 Transient Response of Repeated Poles With Normalization

Taking the inverse of the transform of Eq. (3-16):

$$g(t) = Ae^{-at}\cos\omega t + Ce^{-at}\sin\omega t + x(t) \tag{3-17}$$

In the case of dominant poles, the first two terms will be more effective in the transient response. Hence consider only the first two terms:

$$g'(t) = Ae^{-at}\cos\omega t + Ce^{-at}\sin\omega t \tag{3-18}$$

Rearranging Eq. (3-18):

$$g'(t) = e^{-at}\cos\omega t \left(A + C\frac{\sin\omega t}{\cos\omega t} \right) \tag{3-19}$$

$$g'(t) = Ce^{-at}\cos\omega t \left(\tan\omega t + \frac{A}{C} \right) \tag{3-20}$$

Eq. (3-20) is shown in Fig. 3-3 and its initial value is $g'(0) = A$

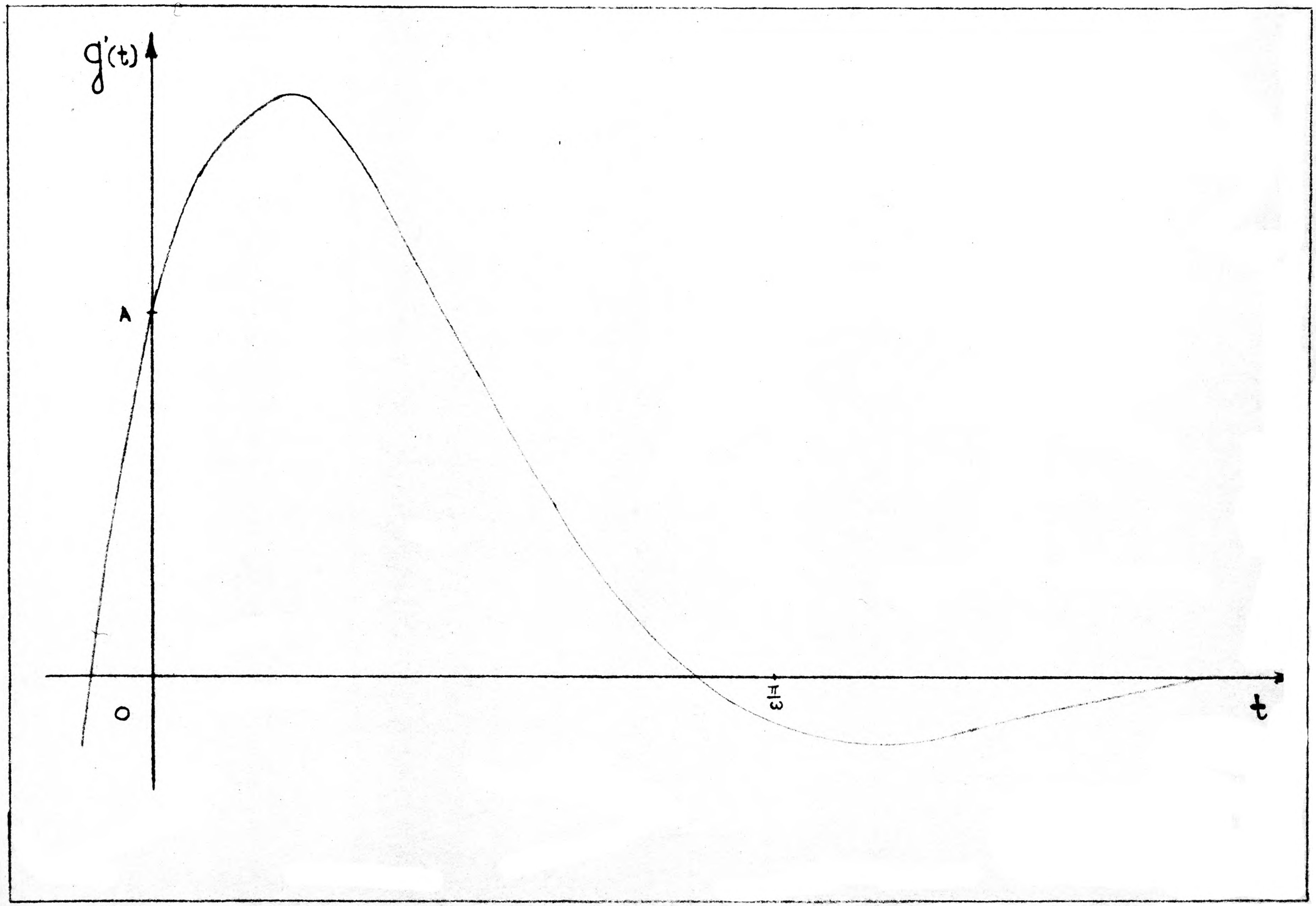

Fig. 3-3 Transient Response Due to Complex Poles

IV. CONSTANT CONTOURS OF A TIME FOR MAXIMUM VALUE

The purpose of this chapter is to develop the mathematical equation for plotting the constant contour curve* on the s-plane, which indicates that the transient response of the complex poles situated on any portion of the curve has maximum value at the same time. The use of scaling is necessary in developing an equation for the constant contour. Therefore a scaling procedure will be given first and then the equation for the constant contour curve will be developed.

A. SCALING

There are two types of scaling: Magnitude Scaling - 'β' and Frequency Scaling - 'α'. One may visualize the scaling process through imagining the s-plane to be a sheet of rubber, whence scaling corresponds to a uniform expansion or contraction in both co-ordinates of the plane. Hence the scaling technique permits designing a system with the right characteristics but the wrong size.

In general, if an arbitrary transfer function is given, then its scaling is as follows (6).

$$\mathcal{L}^{-1}\, \frac{1}{\alpha}\, G\left(\frac{s}{\alpha}\right) = g\,(\alpha t_n) \tag{4-1}$$

Taking a general example whose pole zero configuration is as shown in Fig. 4-1(a) and whose transfer function is given below:

$$G(s) = \frac{K\,(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \pi^2} \tag{4-2}$$

*Constant contour curves can be designated as isochronous curves.

by the inverse Laplace transform Eq. (4-2) will be:

$$g(t) = K e^{-\frac{1}{2}t} \cos \pi t \tag{4-3}$$

The transient response of Eq. (4-3) is shown in Fig. 4-1(b).

Choose the value of scale factor - 'α' such that a pole-zero configuration is as shown in Fig. 4-2(a). Let $\alpha = \frac{1}{2}$, in Eq. (4-1) and substitute Eq. (4-2).

$$\mathcal{L}^{-1} G_s(s_n) = \mathcal{L}^{-1} 2\, G(2s) = \mathcal{L}^{-1} \frac{K(s+\frac{1}{4})}{(s+\frac{1}{4})^2 + (\pi/2)} \tag{4-4}$$

Where $G_s\,(s_n)$ identifies the scaled function and by the inverse Laplace transform, Eq. (4-4) will be:

$$g_s(t_n) = g(\alpha t_n) = K e^{-\frac{1}{4}t_n} \cos \tfrac{1}{2}\pi t_n \tag{4-5}$$

Where $t_n = \frac{t}{\alpha}$. To get the maximum amplitude equal to one, choose $\beta = \frac{1}{k}$, then,

$$g_{s\beta}(t_n) = e^{-\frac{1}{4}t_n} \cos \tfrac{1}{2}\pi t_n \tag{4-6}$$

The transient response of Eq. (4-6) is shown in Fig. 4-2(b).

B. MATHEMATICAL PROOF FOR CONSTANT CONTOUR

When repeated poles are situated at $s = -1$ then their transient response has a maximum value at time, t_n, equal to 1 sec. Now consider the complex poles situated on the imaginary axis at $s = j\frac{\pi}{2}$ (Fig. 4-3(a)), whose transient response is a sinusoidal waveform with period $T = 4$ sec.

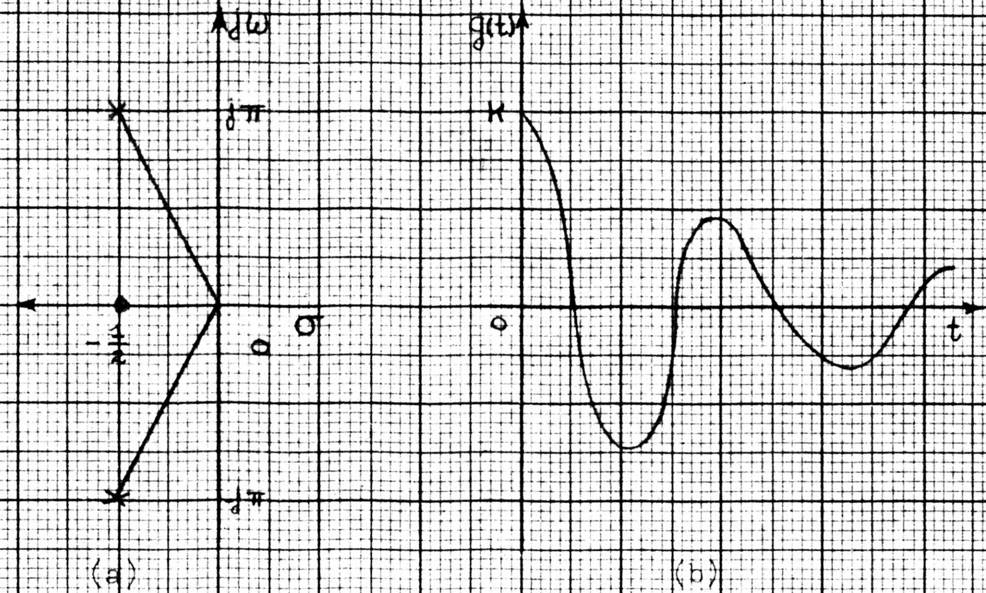

Fig. 4-1 Pole-Zero Configuration and Transient Response for an Unscaled Function.

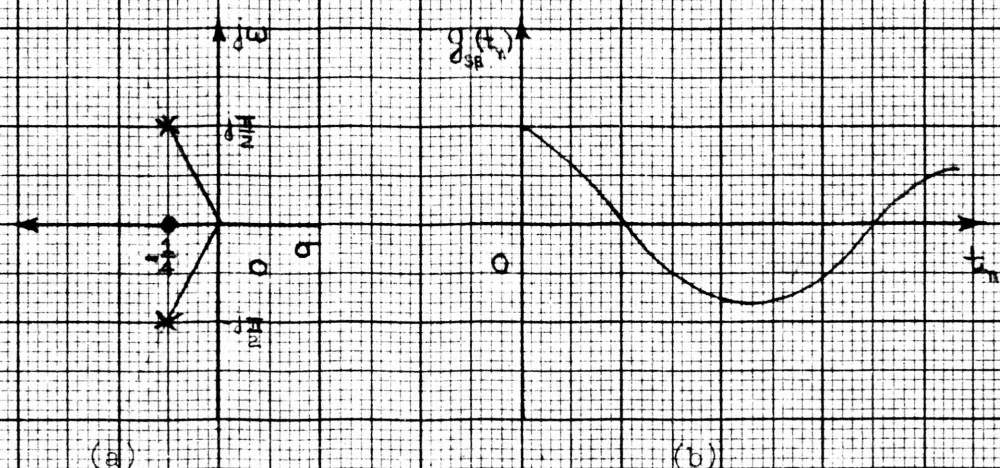

Fig. 4-2 Pole-zero Configuration and Transient Response for A Scaled Function.

and the maximum value is at t = 1 sec. (Fig. 4-3(b). The above two cases are the extreme situation in which one has a damping ratio $\zeta = 1$ while the other has a damping ratio $\zeta = 0$ respectively. Therefore, between the two boundaries it is possible to find the constant contour curve in such a way that the complex poles situated on that contour have a transient response with the maximum value at 1 sec.

Consider the complex poles configuration as shown in Fig. 4-4, whose time function is as follows:

$$g(t) = e^{-at} \sin \omega t \tag{4-7}$$

Now choose normalization factor such as,

$$t_{normalized} = t_n = \frac{t}{\alpha} \tag{4-8}$$

and normalize Eq. (4-7) as shown below:

$$g_s(t_n) = g(\alpha t_n) = e^{-a\alpha t_n} \sin \omega\alpha t_n \tag{4-9}$$

$$\therefore\ g_s(t_n) = e^{-a_s t_n} \sin \omega_s t_n \tag{4-10}$$

Differentiating Eq.(4-10) and equating to zero to get the time, t_n, for maximum value,

$$g_s'(t_n) = e^{-a_s t_n}\left[-a_s \sin \omega_s t_n + \omega_s \cos \omega_s t_n\right] = 0 \tag{4-11}$$

$$\therefore\ a_s \sin \omega_s t_n = \omega_s \cos \omega_s t_n \tag{4-12}$$

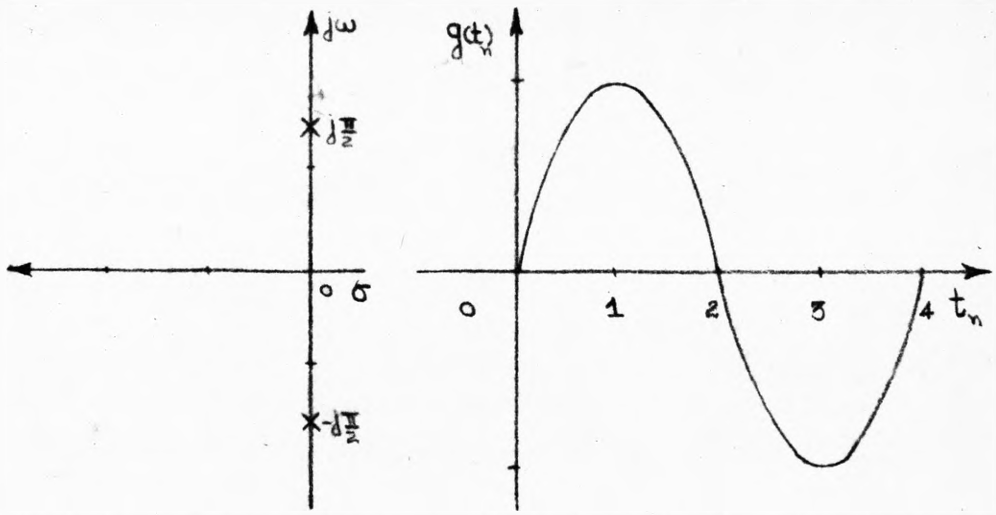

Fig. 4-3(a) Imaginary Poles in the S-Plane

Fig. 4-3(b) Transient Response of Imaginary Poles

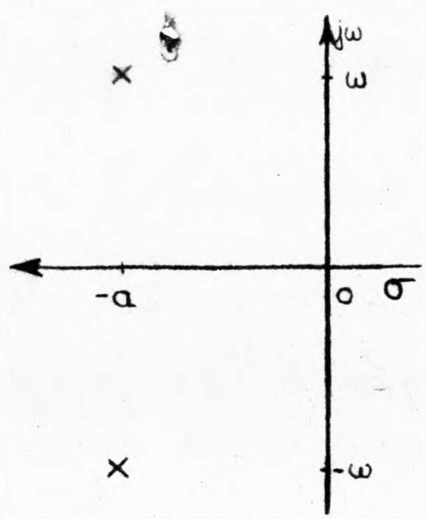

Fig. 4-4 Complex Poles in the S-Plane

$$\therefore \quad \tan \omega_s t_{mn} = \frac{\omega_s}{a_s} \tag{4-13}$$

$$\therefore \quad \tan \omega \frac{t_m}{\alpha} = \frac{\omega_s}{a_s} \tag{4-14}$$

Where t_m = time at peak of the curve.

$\frac{t_m}{\alpha}$ = normalized time at peak.

Choose α such that $\frac{t_m}{\alpha}$ = 1 at peak, then 'a' is a function of φ.
Where φ°(deg.) - $57.3\omega_s$ (rad.)

$$\therefore \quad \tan \omega_s = \frac{\omega_s}{a_s} \tag{4-15}$$

Where ω_s = the imaginary part of the complex pole and a_s = the real part of the complex pole that lies on constant contour curve, UV, whose t_{mn} = 1 sec. The curve, UV, is shown in Fig. 4-5.

$$\therefore \quad \tan \omega_s = \frac{\omega_s}{a(\varphi^\circ)} \qquad \text{OR} \qquad a(\varphi^\circ) = \frac{\omega_s}{\tan \omega_s} \tag{4-16}$$

From Fig. 4-5

$$r = \sqrt{a_s^2 + \omega_s^2} \tag{4-17}$$

substituting Eq. (4-16):

$$r = \sqrt{\frac{\omega_s^2}{\tan^2 \omega_s} + \omega_s^2} \tag{4-18}$$

$$r = \frac{\omega_s}{\tan \omega_s} \sqrt{1 + \tan^2 \omega_s} \tag{4-19}$$

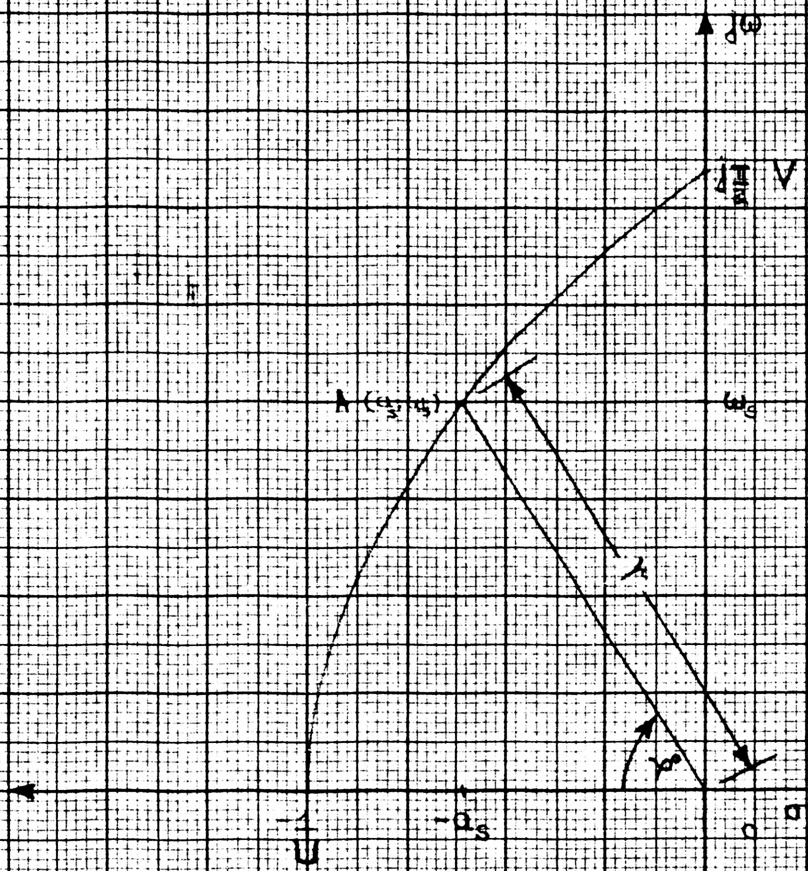

Fig. 4-5 A Constant Contour Curve in the S-Plane

$$\therefore \quad \gamma = \frac{\omega_s}{\tan \omega_s} \cdot \operatorname{Sec} \omega_s \qquad (4\text{-}20)$$

$$\therefore \quad \gamma = \frac{\omega_s}{\operatorname{Sin} \omega_s} \qquad (4\text{-}21)$$

Equation (4-21) is another description which will give the constant contour curve - UV at maximum time, t_{mn}, equal to 1 sec. which is shown in Fig. 4-5. While Fig. 4-6 also shows other curves which have a different time for the maximum value.

C. EQUATION FOR STANDARD PLOT

Let the complex poles be situated anywhere on the s-plane in the transfer function that follows:

$$G(s) = \frac{K \cdot \omega}{(s+a)^2 + \omega^2} \qquad (4\text{-}22)$$

By the inverse Laplace transformation:

$$g(t) = K e^{-at} \sin \omega t \qquad (4\text{-}23)$$

From Fig. 4-6 the value of α is chosen such that the complex poles of Eq.(4-22) lie on the constant contour curve - UV and using Eq. (4-1);

$$\mathcal{L}^{-1} \frac{1}{\alpha} G\left(\frac{s}{\alpha}\right) = \mathcal{L}^{-1} \frac{K \cdot (\omega \alpha)}{(s + a\alpha)^2 + (\omega \alpha)^2} \qquad (4\text{-}24)$$

$$= \mathcal{L}^{-1} \frac{K \cdot \omega_s}{(s + a_s)^2 + \omega_s^2} \qquad (4\text{-}25)$$

TABLE I

ω_s	φ°	$\sin\varphi^\circ$	$\gamma = \frac{\omega_s}{\sin\varphi^\circ}$
0	0°	0.0000	1.0
$\pi/180$	1°	0.01745	1.0
$\pi/36$	5°	0.08716	1.000
$\pi/18$	10°	0.1739	1.001
$\pi/12$	15°	0.259	1.011
$\pi/9$	20°	0.342	1.02
$\pi/7.2$	25°	0.4225	1.031
$\pi/6$	30°	0.500	1.445
$\pi/5.14$	35°	0.574	1.065
$\pi/4.51$	40°	0.644	1.082
$\pi/4$	45°	0.707	1.11
$\pi/3.72$	50°	0.765	1.141
$\pi/3.27$	55°	0.819	1.171
$\pi/3$	60°	0.866	1.21
$\pi/2.77$	65°	0.906	1.251
$\pi/2.57$	70°	0.94	1.3
$\pi/2.4$	75°	0.9659	1.357
$\pi/2.25$	80°	0.985	1.418
$\pi/2.148$	85°	0.996	1.49
$\pi/2.0$	90°	1.001	1.57

Values for Constant Contour Curve-UV

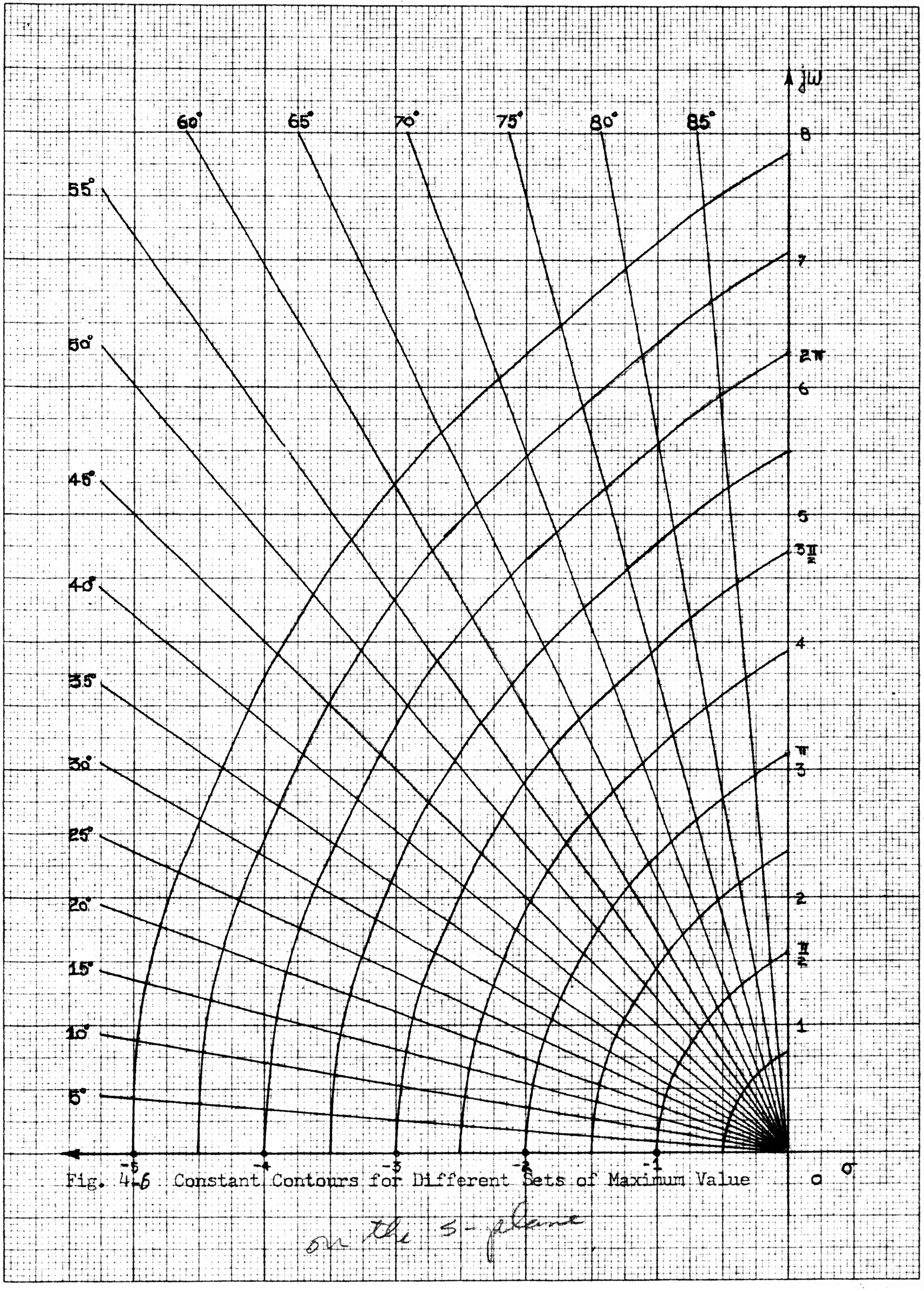

Fig. 4-6 Constant Contours for Different Sets of Maximum Value on the s-plane

From Eq. (4-24):

$$g(\alpha t_n) = K\, e^{-a\alpha t_n} \operatorname{Sin} \omega\alpha t_n \tag{4-26}$$

and from Eq. (4-25):

$$g_s(t_n) = K\, e^{-a_s t_n} \operatorname{Sin} \omega_s t_n \tag{4-27}$$

Eq. (4-27) will give the standard plot with $t_n = 1$ sec. but the maximum amplitude will be different at $t_n = 1$ sec. To obtain the normalized maximum amplitude of 1 the magnitude scaling - 'β' is used. Let $\beta = \frac{e^{a_s}}{K \operatorname{Sin} \omega_s}$, then Eq. (4-27) will be:

$$g_{s\beta}(t_n) = \beta K\, e^{-a_s t_n} \operatorname{Sin} \omega_s t_n \tag{4-28}$$

Now Eq. (4-28) will give the standard plot whose time, t_{mn}, for maximum value is equal to 1 sec. and whose maximum amplitude is also 1.

Fig. 4-7 indicates the standard plots for complex poles with different damping ratios ζ.

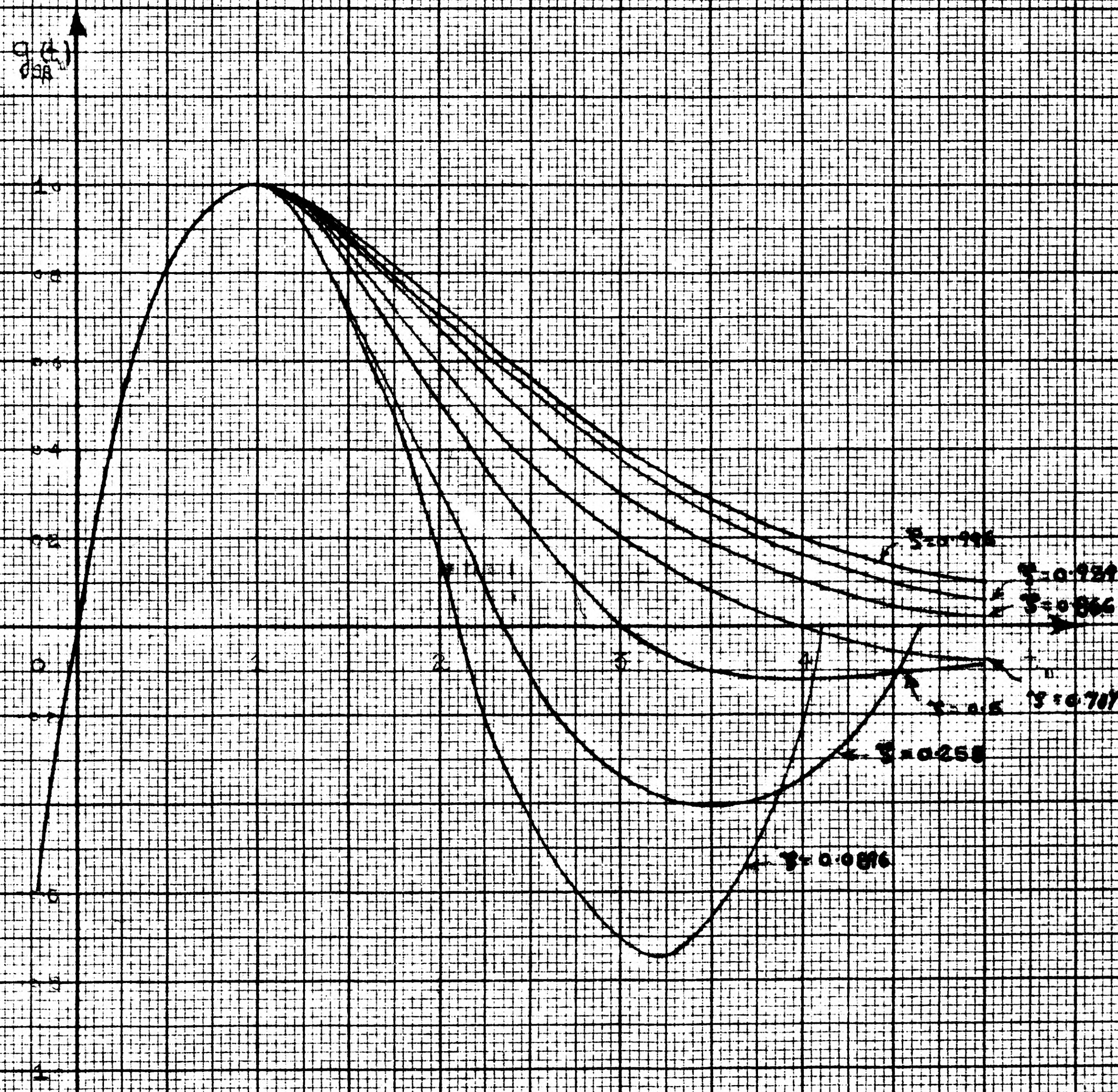

Fig. 4-7 Standard Plots for Complex Poles with Different Damping Ratios.

V. INITIAL AND MAXIMUM VALUES OF COMPLEX POLES

Consider only the transfer function of complex poles as shown in Fig. 5-1:

$$G(s) = \frac{K}{(s+a)^2 + \omega^2} \tag{5-1}$$

$$G(s) = \frac{A's + B'}{(s+a)^2 + \omega^2} \tag{5-2}$$

For complex poles only, Heaviside's coefficient A' is always zero.

$$\therefore\ G(s) = \frac{C \cdot \omega}{(s+a)^2 + \omega^2} \tag{5-3}$$

where B' = Cω

Thus,

$$g(t) = C e^{-at} \sin \omega t \tag{5-4}$$

This transient response is shown in Fig. 5-2. To find the initial value put t = 0 in Eq. (5-4)

$$\therefore\ g(0) = 0 \tag{5-5}$$

To find the time for the maximum value (t_m) the procedure is as follows:

$$g(t) = C e^{-at} \sin \omega t \tag{5-4}$$

Differentiating Eq. (5-4) with respect to time, t, will give:

$$\frac{d}{dt} g(t) = C\, e^{-at} [\omega \cos \omega t - a \sin \omega t] \tag{5-6}$$

Since the maximum value occurs when $\frac{dg(t)}{dt}$ is zero:

$$\therefore \quad \omega \cos \omega t - a \sin \omega t = 0 \tag{5-7}$$

Equating,

$$\tan \omega t_m = \frac{\omega}{a} \tag{5-8}$$

and,

$$t_m = \frac{1}{\omega} \tan^{-1} \frac{\omega}{a} \quad \text{sec} \tag{5-9}$$

Which is shown in Fig. 5-2.

If simple poles and zeros are added to the complex poles of Fig. 5-1 then the transient response associated with the complex poles will be:

$$g'(t) = C\, e^{-at} \cos \omega t \left(\tan \omega t + \frac{A}{C} \right) \tag{3-20}$$

Comparing Eq. (3-20) with that of Eq. (5-4), both are identical only if A in Eq. (3-20) is equal to zero. Therefore, substituting A = 0 into Eq. (3-20)

$$\therefore \quad g'(t) = C\, e^{-at} \cos \omega t \left[\frac{\sin \omega t}{\cos \omega t} + \frac{0}{C} \right] \tag{5-10}$$

$$\therefore \quad g'(t) = C\, e^{-at} \sin \omega t + 0\, e^{-at} \cos \omega t \tag{5-11}$$

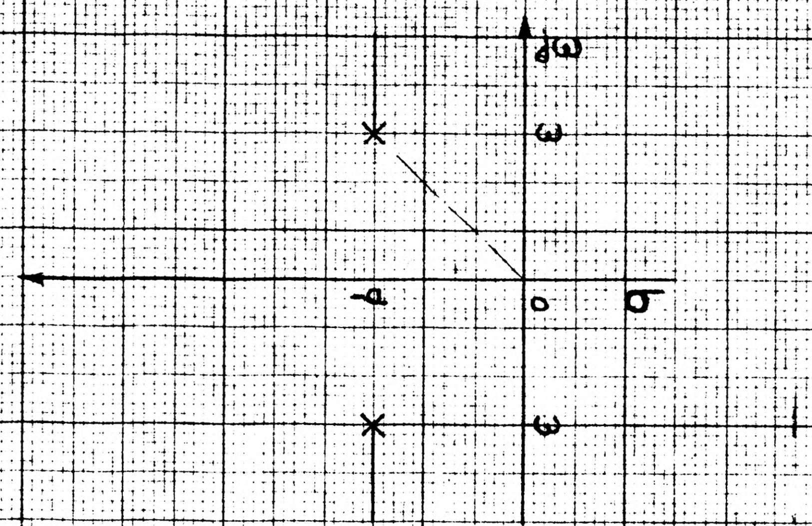

Fig. 5-1 Root Locus Plot of Two Complex Poles

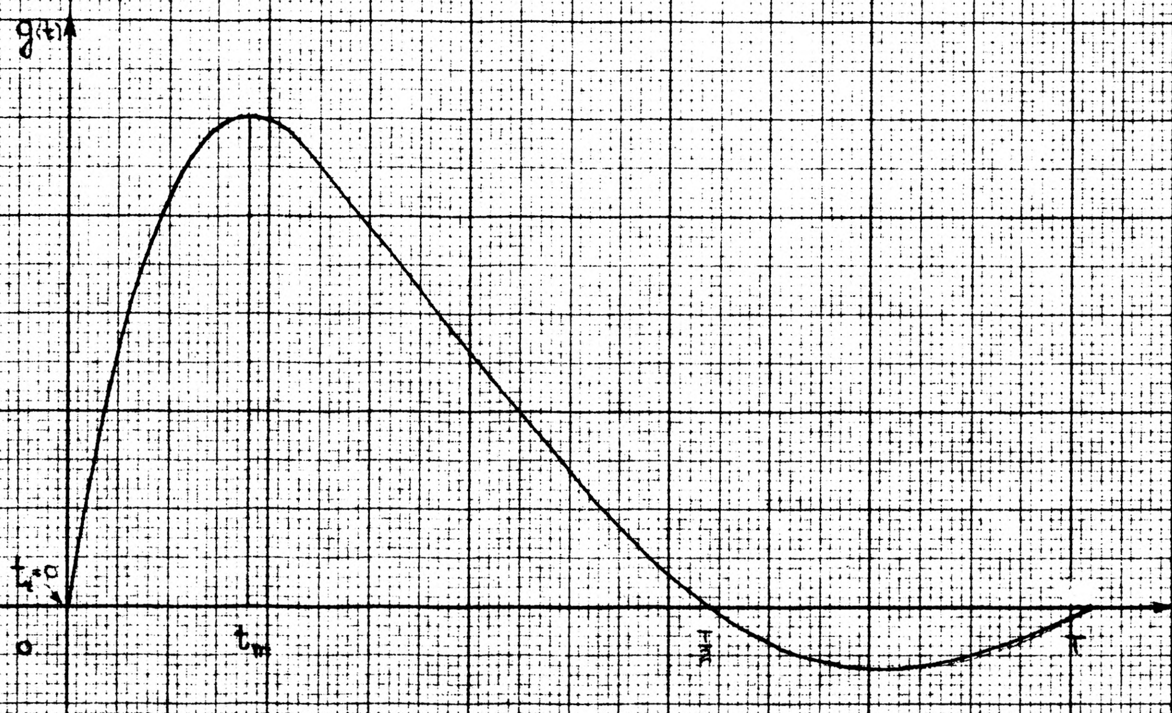

Fig. 5-2 Transient Response of Two Complex Poles.

$$\therefore \quad g'(t) = C\, e^{-at} \sin \omega t \qquad (5\text{-}12)$$

Eq. (5-4) and Eq. (5-12) are identical. Therefore, the addition of a simple pole and zero has changed only the initial value and the transient response associated with the complex poles will be the same.

Consider the transfer function of the pole-zero plot shown in Fig. 5-3:

$$G(s) = \frac{K(s+z_1)(s+z_2)}{s(s+p_1)(s+p_2)\left[(s+a)^2+\omega^2\right]} \qquad (5\text{-}13)$$

$$G(s) = \frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{s+p_2} + \frac{As+K_4}{(s+a)^2+\omega^2} \qquad (5\text{-}14)$$

$$G(s) = \frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{s+p_2} + \frac{A(s+a)}{(s+a)^2+\omega^2} + \frac{C\cdot\omega}{(s+a)^2+\omega^2} \qquad (5\text{-}15)$$

by the inverse Laplace transformation,

$$g(t) = K_1 + K_2 e^{-p_1 t} + K_3 e^{-p_2 t} + C\, e^{-at} \cos \omega t \left(\tan \omega t + \frac{A}{C}\right) \qquad (5\text{-}16)$$

Fig. 5-3 is not in a normalized form. Therefore, by changing a value of α from Fig. 4-6 such that the complex poles lie on the constant contour curve - UV, the associated poles and zeros will shrink according to the value α. A scaled pole-zero configuration is shown in Fig. 5-4. Choosing $t_n = t/\alpha$ and substituting in Eq. (5-16)

$$g(\alpha t_n) = K_1 + K_2 e^{-p_1 \alpha t_n} + K_3 e^{-p_2 \alpha t_n} + C e^{-a \alpha t_n} \cos \omega \alpha t_n \left[\tan \omega \alpha t_n + \frac{A}{C}\right] \qquad (5\text{-}17)$$

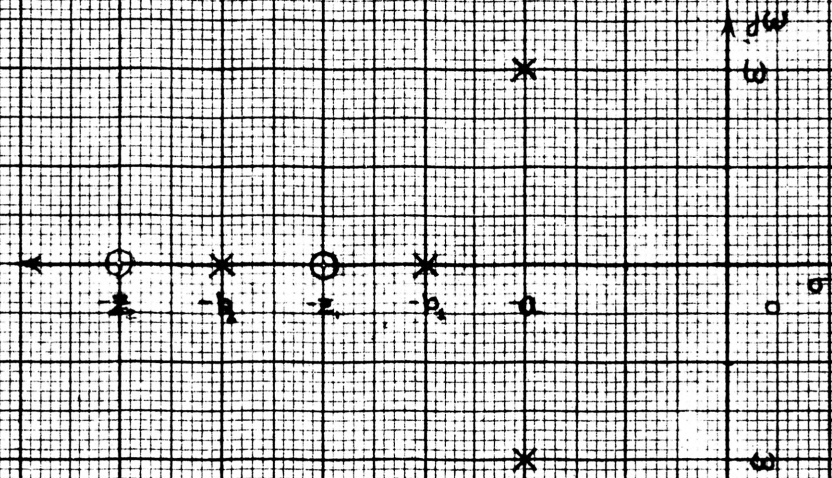

Fig. 5-3 Pole-Zero Configuration for G(s)

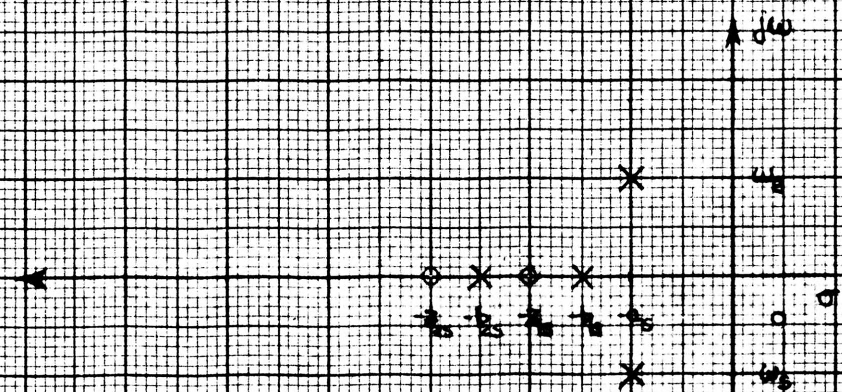

Fig. 5-4 Pole Zero Configuration of $G_n(s)$

$$g_s(t_n) = K_1 + K_2 e^{-p_{1s} t_n} + K_3 e^{-p_{2s} t_n} + C e^{-a_s t_n} \cos \omega_s t_n \left[\tan \omega_s t_n + \frac{A}{C}\right] \qquad (5\text{-}18)$$

Eq. (5-18) is in the normalized form. Only the transient response of Eq. (5-18) associated with the complex poles is needed. Therefore,

$$g_s'(t_n) = C e^{-a_s t_n} \cos \omega_s t_n \left[\tan \omega_s t_n + \frac{A}{C}\right] \qquad (5\text{-}19)$$

The standard plot for the complex poles of Eq. (5-19) is shown in Fig. 5-5. From Eq. (5-19) t_{in} and t_{mn} can be evaluated. The time, t_{in}, for the initial value can be developed by equating,

$$\tan \omega_s t_{in} = \frac{A}{C} \qquad (5\text{-}20)$$

$$\therefore\ t_{in} = \frac{1}{\omega_s} \tan^{-1} \frac{A}{C} \qquad (5\text{-}21)$$

Where t_{in} = the time for initial value in the standard plot as shown in Fig. 5-5

and ω_s = the imaginary part of the normalized complex pole.

If the ratio $\frac{A}{C}$ is equal to the ratio $\frac{\omega}{a}$ then time, t_{mn}, for maximum value can be evaluated as follows and which is shown in Fig. 5-5.

$$\therefore\ \tan \omega_s t_{mn} = \frac{A}{C} = \frac{\omega}{a} \qquad (5\text{-}22)$$

$$\therefore\ t_{mn} = \frac{1}{\omega_s} \tan^{-1} \frac{A}{C} \qquad (5\text{-}23)$$

After evaluating t_{in} for the system, the ratio $\frac{A}{C}$ should be converted to the residue ratio $\frac{A}{b}$ because the residue ratio $\frac{A}{b}$ for complex poles gives the identical relationship with the $\frac{A}{b}$ of the repeated poles. Its mathematical interpretation is as follows:

For repeated poles the transfer function is:

$$G(s) = \frac{K}{(s+a)^2} = \frac{A}{(s+a)} + \frac{ba}{(s+a)^2} \tag{5-24}$$

Now for complex poles the transfer function is:

$$G(s) = \frac{K}{(s+a)^2 + \omega^2} \tag{5-25}$$

$$G(s) = \frac{A(s+a)}{(s+a)^2 + \omega^2} + \frac{C \cdot \omega}{(s+a)^2 + \omega^2} \tag{5-26}$$

Taking the limit as $\omega \to o$ in Eq. (5-26) then:

$$\lim_{\omega \to o} G(s) = \lim_{\omega \to o} \left[\frac{A(s+a)}{(s+a)^2 + \omega^2} + \frac{C \cdot \omega}{(s+a)^2 + \omega^2}\right] \tag{5-27}$$

$$= \frac{A}{s+a} + \frac{C \cdot \omega}{(s+a)^2} \tag{5-28}$$

Equating the residues of Eqs. (5-28) and (5-24) then the result will be:

$$C \cdot \omega = ba \tag{5-29}$$

$$\therefore\ C = b \times \frac{a}{\omega} \tag{5-30}$$

Taking the ratios. Let,

$$\frac{A}{C} = K^* \qquad \text{then} \qquad \frac{A}{b} = K^* \frac{a}{\omega} \tag{5-31}$$

The residue ratio $\frac{A}{b}$ can be used in the standard plot as shown in Fig. 5-5. If the residue ratio $\frac{A}{b}$ for a transfer function of complex poles is 1, this would indicate the maximum value corresponding to the time, t_{mn}, equal to 1 sec. as shown in Fig. 5-5. If $\frac{A}{b}$ for a transfer function is greater than 1, then the initial point will be to the right of $\frac{A}{b} = 1$. However, if $\frac{A}{b}$ is less than 1 then the initial point will be to the left of $\frac{A}{b} = 1$.

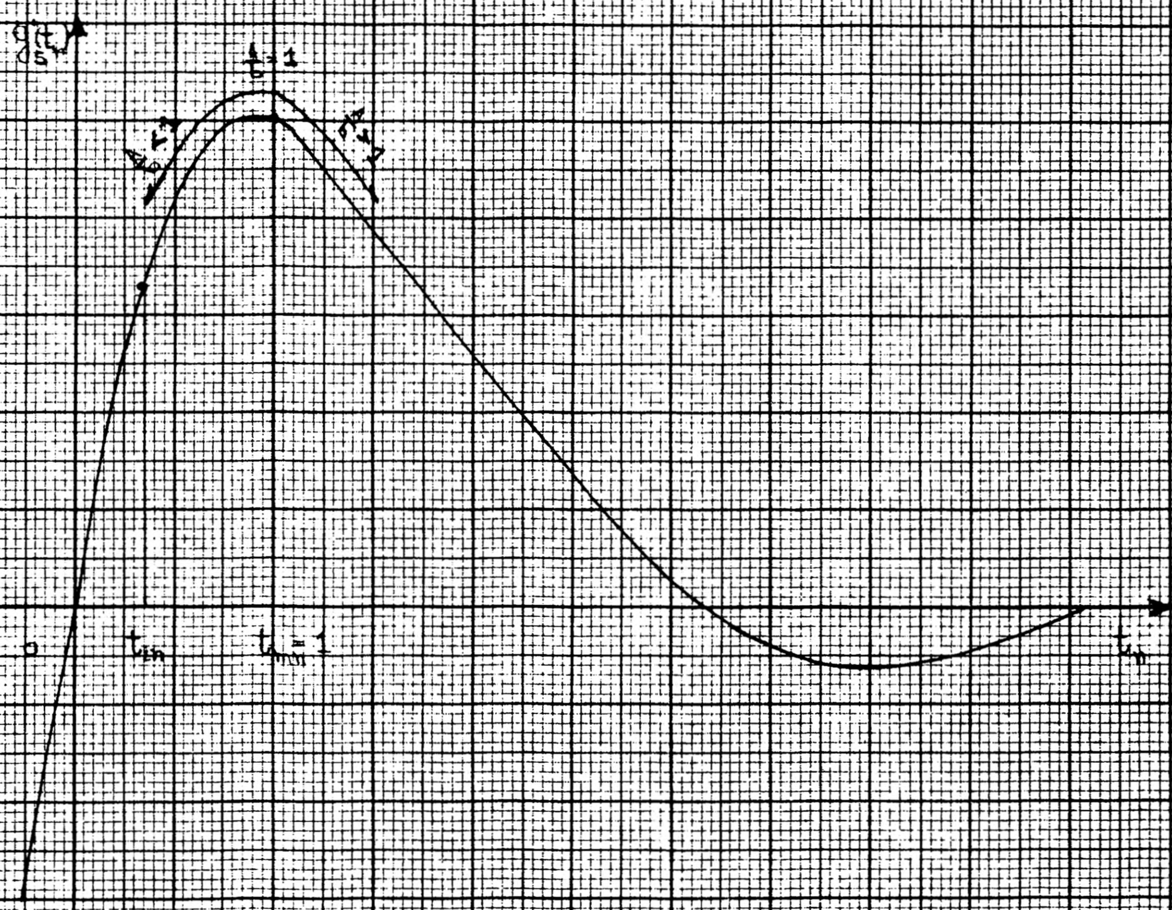

Fig. 5-5 Standard Plot for the Complex Poles of Eq. 5-19.

VI. THE RESIDUE RATIO METHOD FOR CALCULATING THE INITIAL VALUE OF A TRANSIENT RESPONSE FOR COMPLEX POLES

Consider the transfer function:

$$G_n(s) = \frac{K\,(s+Z_{1s})\,(s+Z_{2s})}{s\,(s+P_{1s})(s+P_{2s})\,[(s+\alpha_s)^2+\omega_s^2]} \tag{6-1}$$

which can be arranged as:

$$G_n(s) = \frac{K\,N(s)}{[(s+\alpha_s)^2+\omega_s^2]\,D(s)} \tag{6-2}$$

By partial fraction expansion:

$$G_n(s) = \frac{k}{s+\alpha_s-j\omega_s} + \frac{k^*}{s+\alpha_s+j\omega_s} + \frac{N'(s)}{D(s)} \tag{6-3}$$

From Eq. 6-3 or graphically find the value of the residue, k,

let, $k = P+jQ$ and $k^* = P-jQ$

or, $$k = m\,e^{-j\phi} \tag{6-4}$$

Put the value of residue, k, on the complex pole, $X = \alpha_s + j\omega_s$ shown in Fig. 6-1. The values obtained from the Fig. 6-1 are shown below

$$P = XF \qquad \text{and} \qquad Q = FY \tag{6-5}$$

and $XY = me^{-j\phi}$

Cotangent of the angle YXF is the ratio $\frac{A}{C}$, or the tangent of the angle XYF is the ratio $\frac{A}{C}$.

A geometrical interpretation can be used to evaluate the ratio and will be useful in later derivations.

(a) Take the original transfer function and by means of the root locus technique determine the angle of emergence i.e., GXE.

(b) The angle of emergence will be exactly 180° out of phase with respect to residue, k, i.e., XY in polar form.

(c) A perpendicular XD is drawn with respect to line EXY. Point D is on the negative real axis.

From Fig. 6-1:

$$\angle XFY = \angle XHD = 90^{\circ} \tag{6-6}$$

and

$$\angle XYF = \angle FXD = \angle XDH \tag{6-7}$$

Thus,

$$\text{Tangent } \angle XDH = \tan \omega_s t_{in} = \frac{A}{C} = K^* \tag{6-8}$$

Where DH is a positive vector and XH is a positive vector. Hence the ratio $\frac{A}{C}$ is positive. From Eq.(6-8), t_{in}, can be evaluated. Substitute the value $C = b\frac{a_s}{\omega_s}$ in Eq. (6-8). Therefore the residue ratio,

$$\frac{A}{b} = K^* \frac{a_s}{\omega_s} \tag{6-9}$$

It is also possible to evaluate the residue ratio $\frac{A}{b}$ for the complex poles by a graphical method. Fig. 6-1 shows that:

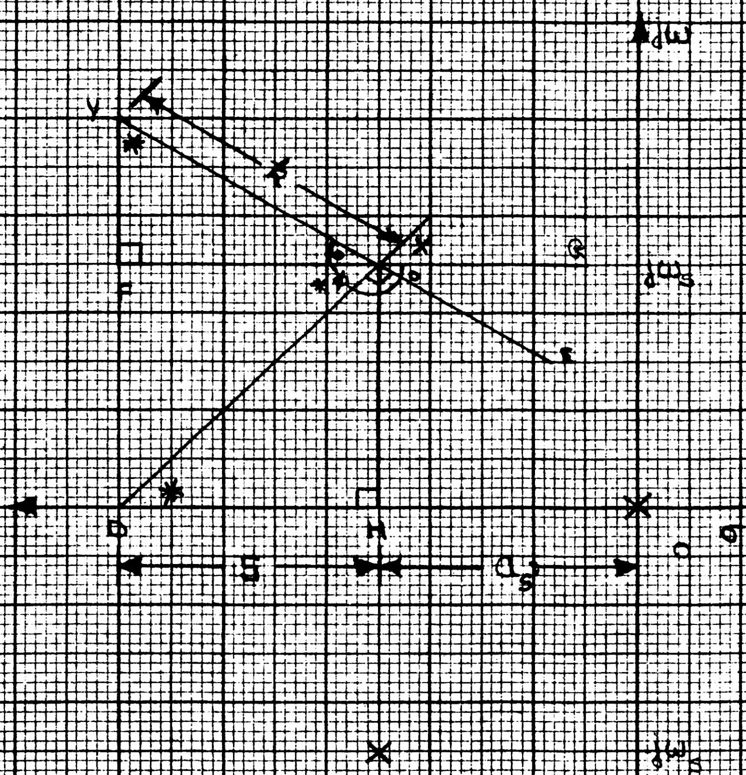

Fig. 6-1 Geometrical Method of Evaluating the Residue Ratio in the S-Plane.

$a_s =$ OH the real part of the complex pole.

$\omega_s =$ XH the imaginary part of the complex pole.

From Eq. (6-8):

$$\frac{K^*}{\omega_s} = \frac{\tan \omega_s t_{in}}{\omega_s} = \frac{XH}{DH} \times \frac{1}{XH} = \frac{1}{DH} \qquad (6\text{-}10)$$

Now taking Eq. (6-9)

$$\frac{A}{b} = K^* \frac{a_s}{\omega_s} = \frac{a_s}{S} \qquad (6\text{-}11)$$

Therefore $\frac{A}{b}$ can be determined by this graphical method. The residue ratio $\frac{A}{b}$ can be positive or negative depending on the sign of line DH. If the vector DH is to the right from the point D then the sign is positive and if it is to the left then it is negative.

A brief procedure is given for the evaluation of t_{in} from a general transfer function. From the transfer function the angle of emergence is first evaluated. Scale the transfer function such that the complex poles lie on the constant contour curve - UV and ω_s is then determined. The time, t_{in} for the initial value is calculated by applying Eq. (6-8) and the residue ratio $\frac{A}{b}$ is evaluated from Eq. (6-9). Several examples are given below.

Example 1

The transfer function shown in Fig. 6-2(a_1), is given as follows:

$$G(s) = \frac{1(s+1)}{s[(s+1)^2+1]} \qquad (6\text{-}12)$$

The angle of emergence $= +45^\circ$ is shown in Fig. 6-2(a_1).

The value of $\alpha = 0.785$ is determined from Fig. 4-6,

and $\tan \omega_s t_{in} = -1$ where $\omega_s = \omega\alpha = 1 \times .785 = .785$

$$\therefore\ t_{in} = -\frac{1}{\omega_s}\tan^{-1}1 = -\frac{45^\circ \times .01745}{.785} = -1 \text{ Sec}$$

The residue ratio is shown in Fig. 6-2(b_1) and given by

$$\frac{A}{b} = -1 \times \frac{a}{\omega} = -1 \qquad \text{OR} \qquad \frac{A}{b} = \frac{a}{5} = -1$$

<u>Example 2</u>

The transfer function, shown in Fig. 6-2(a_2), is given as follows:

$$G(s) = \frac{192}{s(s+3)(s^2+6s+64)} \tag{6-13}$$

Angle of emergence = -112.1° which is shown in Fig, 6-2(a_2).

The value of $\alpha = 0.1921$ is determined from Fig. 4-6.

Therefore,

$$\tan \omega_s t_{in} = -.406 \qquad \text{where} \quad \omega_s = \omega \cdot \alpha = 7.4 \times .192 = 1.42$$

$$t_{in} = -\frac{1}{\omega_s}\tan^{-1}0.406 = -\frac{.386}{1.42} = -.272 \text{ Sec.}$$

The residue ratio, is shown in Fig. 6-2(b_2) and given by

$$\frac{A}{b} = -\frac{.406}{7.4} \times 3 = -0.1645$$

<u>Example 3</u>

The transfer function, shown in Fig. 6-2(a_3) is given as follows:

$$G(s) = \frac{1}{s(s+2)(s^2+2s+2)} \tag{6-14}$$

The angle of emergence $= -90^\circ$ is shown in Fig. 6-2 (a_3).

The value of $\alpha = 0.785$ is determined from Fig. 4-6.

Therefore,

$\tan \omega_s t_{in} = 0$ where $\omega_s = \omega \cdot \alpha = 1 \times .785$

$t_{in} = 0$

The residue ratio is shown in Fig. 6-2(b_3) and given by:

$\frac{A}{b} = 0$

Example 4

The transfer function, shown in Fig. 6-2(a_4) is given as follows:

$$G(s) = \frac{7.75}{s(s+3)(s^2+2s+2.25)} \qquad (6\text{-}15)$$

The angle of emergence = -70.75 is shown in Fig. 6-2(a_4).

The value of α = 0.656 is determined from Fig. 4-6.

Therefore,

$\tan \omega_s t_{in} = +.354$ where $\omega_s = .656 \times 1.5 = .984$

$t_{in} = \frac{1}{\omega_s} \tan^{-1} .354 = .346$

The residue ratio is shown in Fig. 6-2(b_4) and given by:

$\frac{A}{b} = \frac{.354}{1.5} \times 1 = .236$

Example 5

The transfer function shown in Fig. 6-2(a_5) is given as follows:

$$G(s) = \frac{1.25}{s\left[(s+1)^2 + (0.5)^2\right]} \qquad (6\text{-}16)$$

The angle of emergence = -63.4° is shown in Fig. 6-2(a_5).

The value of α = 0.92 is determined from Fig. 4-6.

Therefore,

$\tan \omega_s t_{in} = 0.5$ where $\omega_s = 0.5 \times 0.92 = .46$

$t_{in} = \frac{1}{\omega_s} \tan^{-1} 0.5 = \frac{26.6^{\circ} \times .01745}{0.46} = 1$ sec.

The residue ratio is shown in Fig. 6-2(b_5) and given by:

$\frac{A}{b} = \frac{.5 \times 1}{.5} = 1$

Example 6

The transfer function, shown in Fig. 6-2(a_6) is given as follows:

$$G(s) = \frac{1\,(s+3)}{s\left[(s+1)^2+1\right]} \tag{6-17}$$

The angle of emergence $= -18.4^{\circ}$ is shown in Fig. 6-2(a_6).

The value of $\alpha = 0.785$.

Therefore,

$$\tan \omega_s t_{in} = 3 \qquad \text{where} \quad \omega_s = .785 \times 1 = .785$$

$$t_{in} = \frac{1}{\omega_s} \tan^{-1} 3 = 1.59$$

The residue ratio is shown in Fig. 6-2(b_6) and given by:

$$\frac{A}{b} = 3$$

Example 7

The transfer function shown in Fig. 6-2(a_7) is given as follows:

$$G(s) = \frac{(s+2)}{s\left[(s+1)^2+1\right]} \tag{6-18}$$

The angle of emergence $= 0^{\circ}$ is shown in Fig. 6-2(a_7).

The value of $\alpha = 0.785$.

Therefore,

$$\tan \omega_s t_{in} = \infty \qquad \text{where} \quad \omega_s = 0.785 \times 1 = .785$$

$$t_{in} = \frac{1}{\omega_s} \tan^{-1} \infty = \frac{\pi}{2} \times \frac{1}{.785} = \qquad \text{sec}$$

The residue ratio is shown in Fig. 6-2(b_7) and given by:

$$\frac{A}{b} = \infty$$

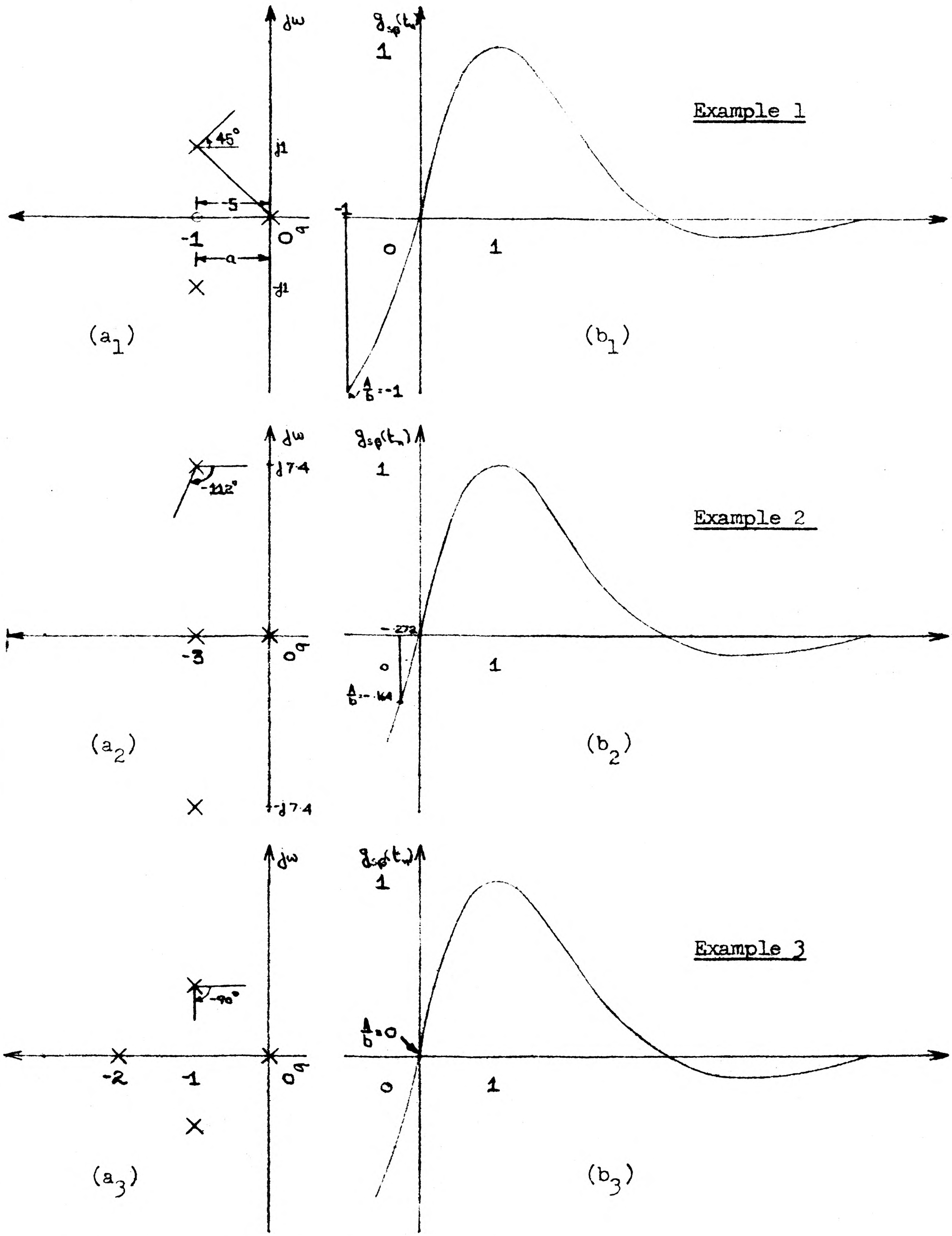

Fig. 6-2 Examples for Complex Poles.

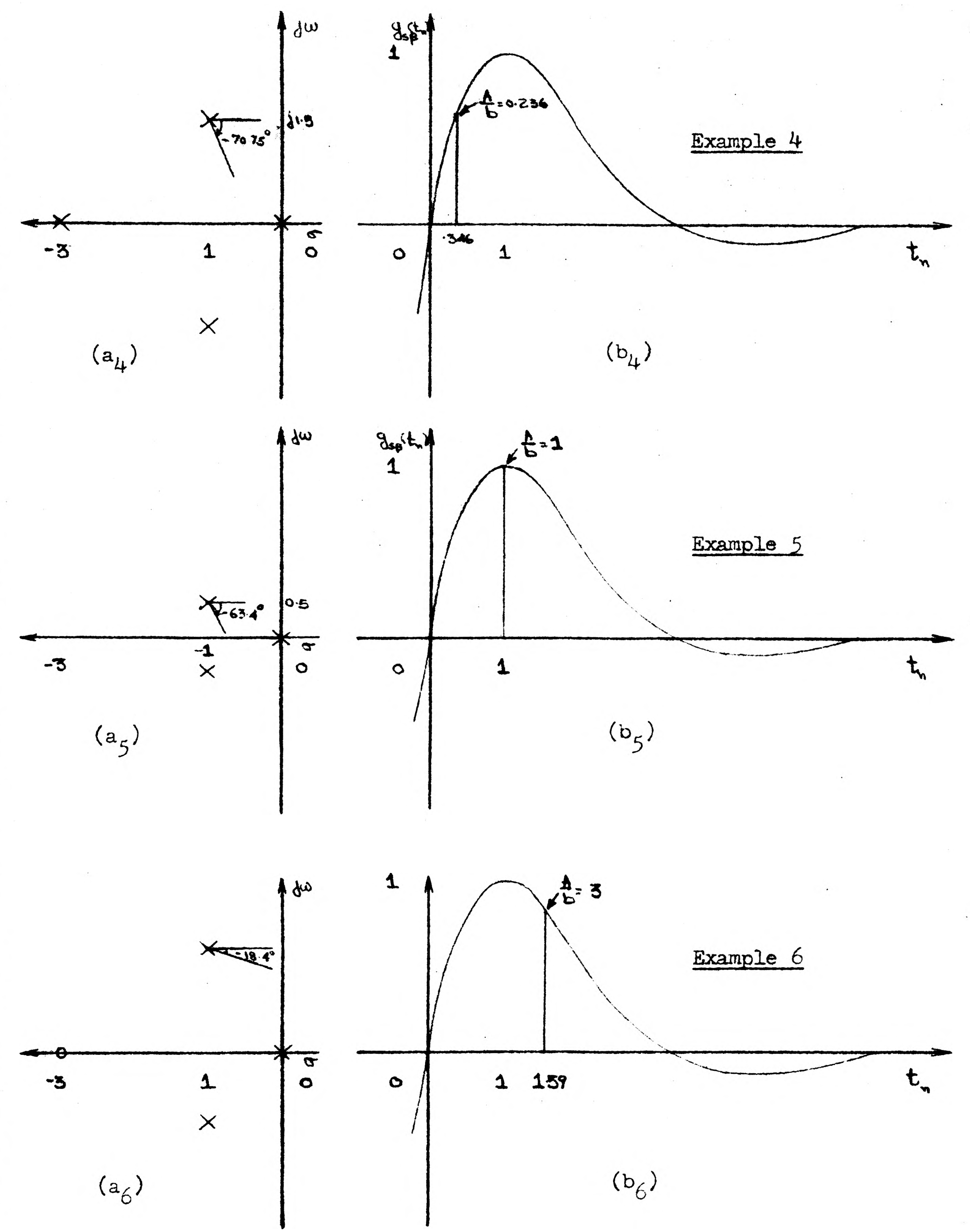

Fig. 6-2 Examples for Complex Poles

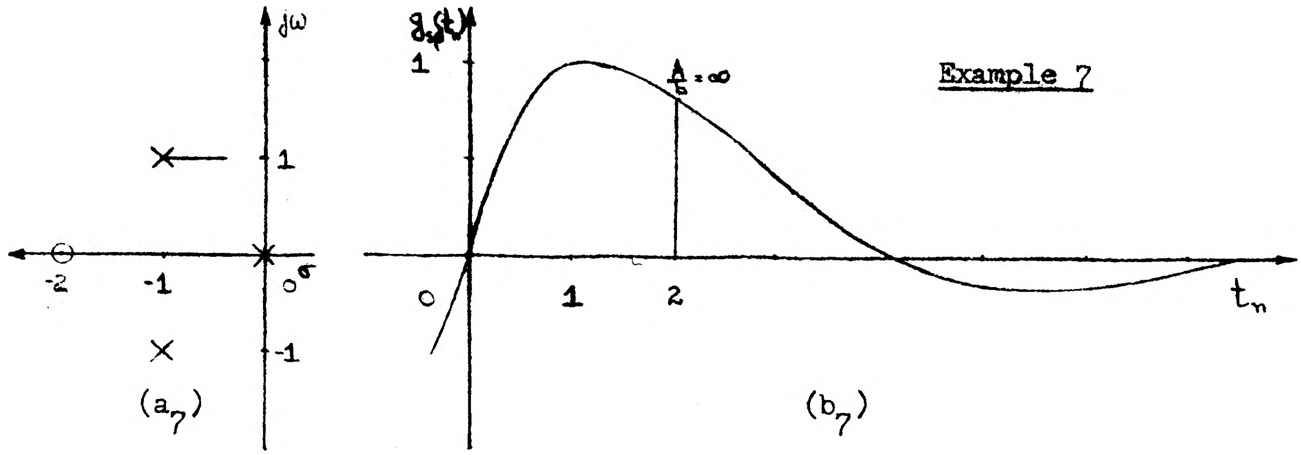

Fig. 6-2 Examples for Complex Poles

VII. RESIDUE RATIO METHOD FOR CALCULATING THE INITIAL VALUE OF A TRANSIENT RESPONSE FOR REPEATED POLES

The object of this chapter is to develop an equation for determining the residue ratio quickly for repeated poles. The radius of curvature is obtained graphically from the root locus plot and is used for determing the residue ratio.

The root locus plot of a function having repeated poles will produce a locus having an angle of emergence of $\pm 90^\circ$ on the negative real axis. Therefore, it is not possible to obtain any angle with the pole and obtain an exact residue ratio. The root locus of a special transfer-function with a simple zero and complex poles is shown in Fig. 7-1(a), an approximate method for determining the radius of curvature S' is given. Let the transfer function be given as:

$$G(s) = \frac{K(s+z_1)}{(s+a)^2+\omega^2} \tag{7-1}$$

$$G(s) = \frac{A(s+a)}{(s+a)^2+\omega^2} + \frac{C\cdot\omega}{(s+a)^2+\omega^2} \tag{7-2}$$

The residue ratio, $\frac{A}{b}$ can be evaluated as shown in Fig. 7-1(a) which is given as $\frac{A}{b} = \frac{a}{S}$.

Consider the limiting case as the complex poles approach the negative real axis on the root locus plot of Fig. 7-1(a). The frequency ω tends to zero and the perpendicular line XH will be small as compared to the lines XD and DH. In the present case, the complex pole is situated at X', adjacent to the negative real axis. The segment X' H' is small and approximately tangent to the root locus.

Therefore DX'$\approx$ DH' and X'H' = $\omega' \approx 0$ (7-3)

and $$\frac{A}{b} = \frac{a'}{DH'} \approx \frac{a'}{DX'} = \frac{a'}{S'} \qquad (7\text{-}4)$$

Eq. (7-4) indicates that instead of using projection DH', the radius of curvature DX' can be used to evaluate the residue ratio.

The root locus plot shown in Fig. 7-1 (b) has repeated poles situated at point H'. The residue ratio, $\frac{A}{b}$ is evaluated as follows:

(a) Determine a' which is the known distance from the repeated poles H' to the origin.

(b) Evaluate S' by finding the radius of curvature at the repeated poles H'. This is accomplished by use of dividers. Choose the center point, D, in such a way that radius of curvature, S' is tangent to a greater part of the root locus near the point, H'. In this case, the root locus is a circle, therefore, S' fits almost all of the circular part of the root locus. Then:

$$\frac{A}{b} = \frac{a'}{S'} \qquad (7\text{-}5)$$

Where a' = the distance from the origin to the second order pole, and S' = the radius of curvature of the root locus plot at the second order pole. If the center of curvature is to the left of the second order pole, S' is positive. If the center of curvature is to the right, S' is negative.

This method for determination of radius of curvature - S' is approx-

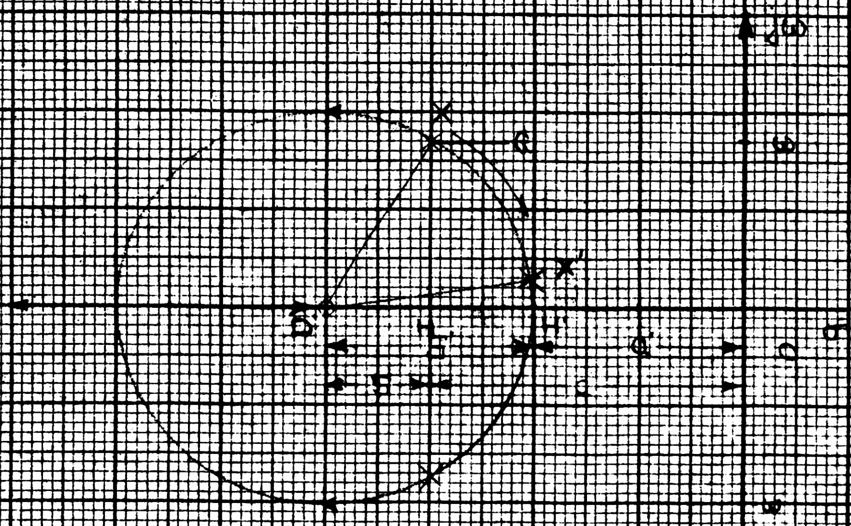

Fig. 7-1a Root Locus Diagram of G(s)

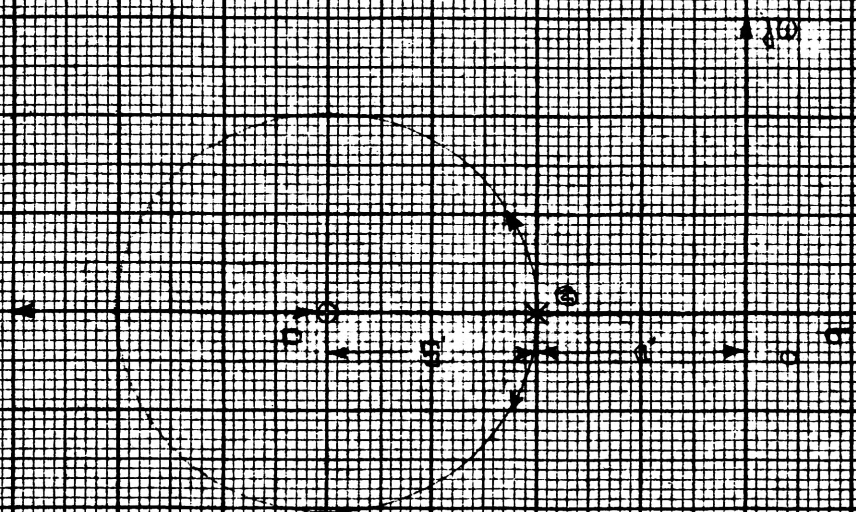

Fig. 7-1b Root Locus Diagram of a Simple Zero and Repeated Poles

imate for two reasons: (1) the root locus plotted for repeated poles is in itself approximate, and (2) the measurement of S' by means of the divider is also approximate.

Several examples for repeated poles are given below:

Example 1

The transfer function shown in Fig. 7-2(a_1) is given as:

$$G(s) = \frac{1}{(s+2)(s+1)^2} \tag{7-6}$$

The value of the radius of curvature - S is equal to -1.

Therefore,

$\frac{A}{b} = \frac{a}{S} = \frac{1}{-1} = -1.$ This is shown in Fig. 7-2(b_1)

Example 2

The transfer function shown in Fig. 7-2(a_2) is given as:

$$G(s) = \frac{1(s+2)}{(s+1)^2} \tag{7-7}$$

The value of the radius of curvature -S is equal to 1.

Therefore,

$\frac{A}{b} = \frac{a}{S} = \frac{1}{+1} = +1.$ This is shown in Fig. 7-2(b_2).

Example 3

The transfer function shown in Fig. 7-2(a_3) is given as:

$$G(s) = \frac{23\,(s+3)}{(s+1.15)(s+9)^2} \tag{7-8}$$

The value of the radius of curvature $-S$ is equal to 25.

Therefore,

$$\frac{A}{b} = \frac{a}{S} = \frac{9.00}{25} = .359.$$ This is shown in Fig. 7-2(b_3)

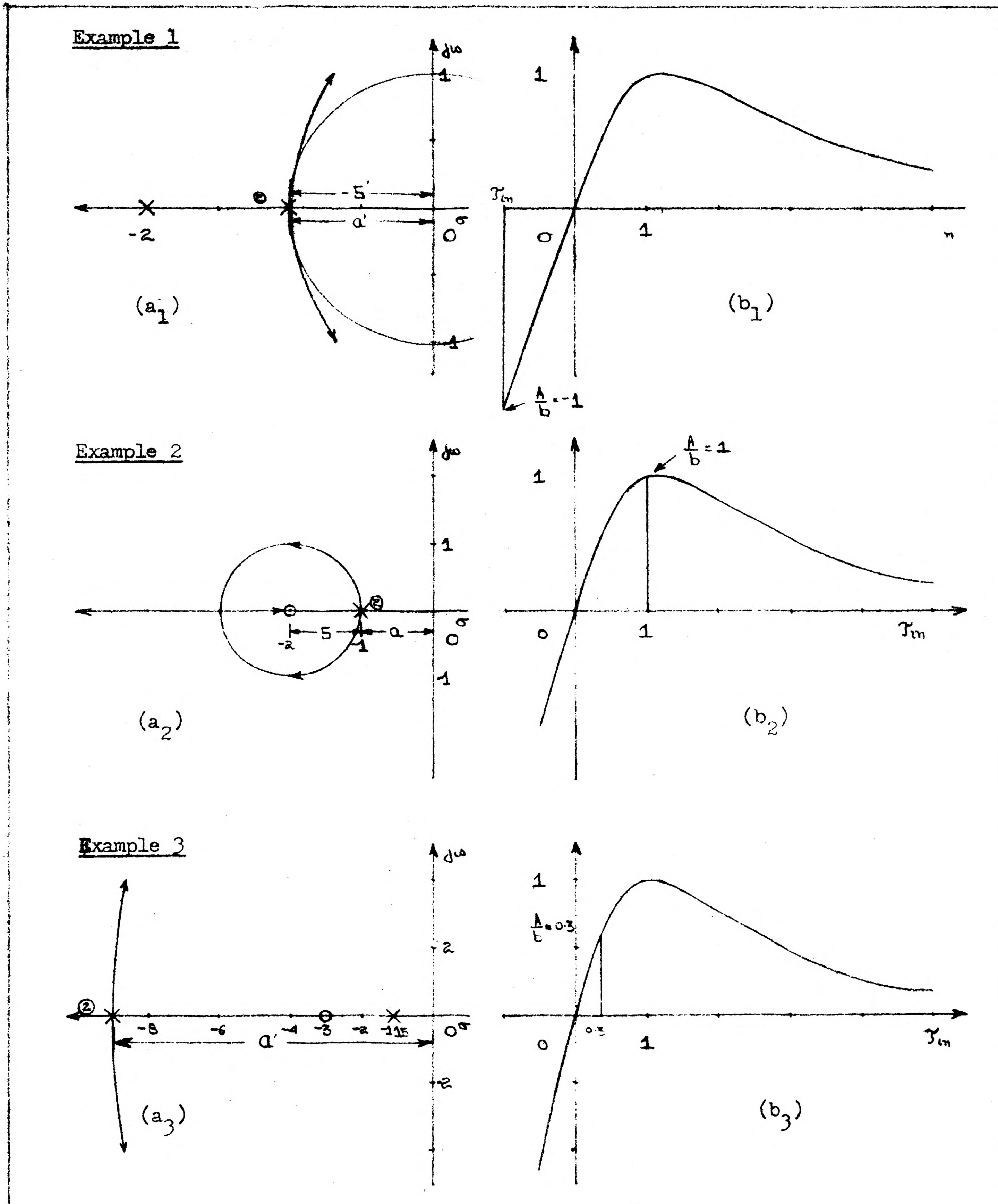

Fig. 7-2 Examples for Repeated Poles

VIII. CONCLUSIONS

The transient response of any system is primarily due to the dominant poles in the s-plane. While the other simple singularities will just increase or decrease the speed of response. If the dominant poles are complex conjugates, then the approximate transient response will be due to these poles and the residue ratio in conjuction with a standard plot will give the initial value and the time for maximum value. The residue ratio of the complex poles, or of repeated poles will be affected by the other singularities in the s-plane.

Standard plots for the complex and repeated poles have been plotted such that their maximum value occurs at time, t_n, equal to 1 second. Hence the approximate transient response of the system can be determined quickly.

The residue-ratio method for complex poles is exact, but for repeated poles, the method is approximate because of the graphical procedure, and the assumption that the response of the complex poles on the root locus near the repeated poles is similar. The main purpose of the residue-ratio method is to obtain a way to determine the initial value and the approximate response of the system quickly.

BIBLIOGRAPHY

1. TRUXAL, J. G. (Book) Automatic Feedback Control System Synthesis. McGraw-Hill Book Company, Inc., New York.

2. SKILLING, H. H. (Book) Transient Electric Currents. McGraw-Hill Book Company, Inc., New York.

3. MULLIGAN, J. H., Jr. (1949) The Effect of Pole and Zero Locations on the Transient Response of Linear Dynamic Systems. Proceedings of the I.R.E.(1957) pp. 516-529.

4. KUO, Benjamin C. (Book) Automatic Control Systems. Prentice Hall, New Jersey.

5. VAN VALKENBURG, M. E. (Book) Network Analysis. Prentice Hall, Inc., New Jersey.

6. VAN VALKENBURG, M. E. (Book) Modern Network Synthesis. John Wiley & Sons, Inc., New York - London.

VITA

The author, Rameshchandra Shankerbhai Patel, was born on November 1, 1937 in Ahmedabad, India. He received his primary education at Chhotaundepur, India and his secondary education in Ahmedabad, India. He received the Diploma in Mechanical Engineering in 1958; and the Diploma in Electrical Engineering in 1959 from L. D. College of Engineering, Ahmedabad, India. He received the B.S. Degree in Electrical Engineering from the Missouri School of Mines and Metallurgy, Rolla, Missouri in 1961.

He enrolled in the Department of Electrical Engineering, Missouri School of Mines and Metallurgy, for graduate study in January, 1961.